Automorphism Groups of Partially Ordered Sets

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Abstract

In 2020, Johnathan Barmak [1] proved that every finite group G is the automorphism group of a poset P with $4|G|$ points. In his paper, he also discusses the least positive number of points in a poset P needed to realize a group G, denoted as $\beta(G)$. For our research, we proved that every finitely generated abelian group is the automorphism group of some poset. We also investigated the $\beta(S_n)$ and $\beta(Z_{p^k})$. Lastly, we developed code that determines the automorphism group of a given finite poset.

History

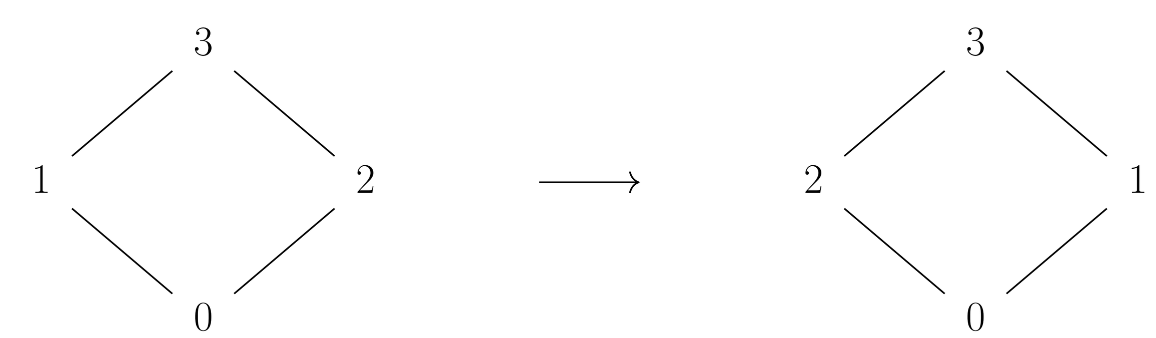
Predating Barmak's 2020 proof [1], Barmak and Minian [2] proved in 2009 that any finite group G of order n with r generators can be realized as the automorphism group of some poset P with $n(r+2)$ points. Unknown to them, Robert Frucht [4], a German mathematician proved this same theorem in 1939, who's paper wasn't translated into English until 1949. Barmak and Minian referenced two authors, Birkhoff and Thornton, who presumably also did not know about Frucht's paper. Birkhoff [3] proved in 1946 that all finite groups G are the automorphism of some poset P. Thornton [5] proved in 1972 any finite group G of order n and r generators can be realized with a poset of $n(2r+1)$ elements.

Definitions

- A **poset** $(X, \leq_X)$ is a set X with a relation $\leq_X$ such that for all $x, y, z \in X$, we have $x \leq_X x$; $x \leq_X y$ and $y \leq_X x \implies y = x$; and $x \leq_X y$ and $y \leq_X z \implies x \leq_X z$.
- A poset P is **connected** if for all $x, y \in P$ there exists a sequence $(x = \alpha_1, \alpha_2, \dots, \alpha_n = y)$ of points $\alpha_i \in P$ such that each α_i is comparable to α_{i+1}.
- Let $x, y \in P$. We say y *covers* x if $y > x$ and if $z \in P$ is such that $x \leq z \leq y$, then $z = x$ or $z = y$.
- We say $f : P \to Q$ is a **poset map** if for all $x, y \in P$, we have $x \leq_P y \implies f(x) \leq_Q f(y)$.
- A poset map $f : P \to Q$ is said to be **order-reflexive** if for all $x, y \in P$, whenever $f(x) \leq_Q f(y)$, we have $x \leq_P y$.
- A surjective, order-reflexive map $f : P \to Q$ is called a **poset isomorphism**.
- An isomorphism $f : P \to P$ is called an **automorphism**. The set $\operatorname{Aut} P$ of all automorphisms from P onto P is a group under composition.
- A group G is **realizable** if there exists a poset P such that $\operatorname{Aut} P = G$.
- $\boldsymbol{\beta(G)} = \min\{|P| : P \text{ is a poset with } \operatorname{Aut} P \cong G\}$.

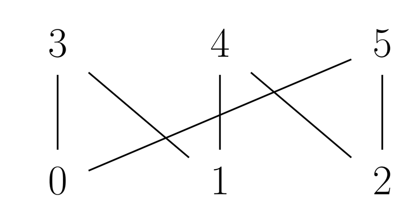
Automorphisms

Consider the following poset P and its automorphism:



Of all surjective maps $P \to P$, only 2 are automorphisms. $\operatorname{Aut} P \cong \mathbb{Z}_2$.

Consider the following poset Q:

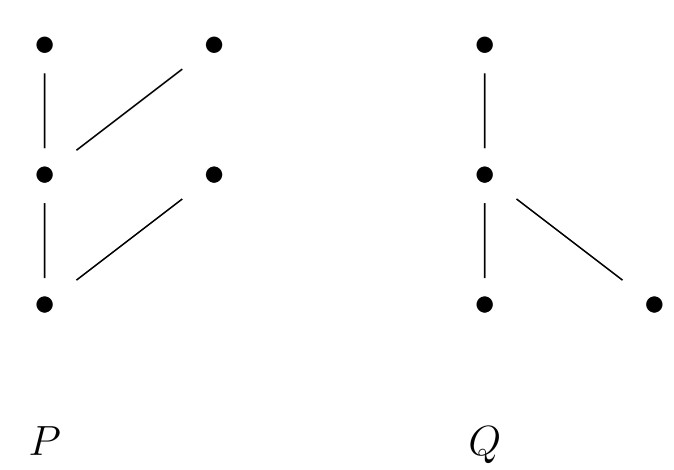


Of all surjective maps $Q \to Q$, only 6 are automorphisms. $\operatorname{Aut} Q \cong \mathbb{Z}$

Disjoint Union Theorem

If P and Q are posets with disjoint sets of points, the poset $R := P \sqcup Q$ is defined to have the order $x \leq_R y$ if and only if $x \leq_P y$ or $x \leq_Q y$.

In the disjoint union, no elements of P are comparable to any elements of Q. The Hasse diagram for the disjoint union looks like P and Q put next to each other.



If $\mathcal{P} = \sqcup_{i=1}^{k} P_i$ where each P_i is a connected poset, and no two distinct P_i are isomorphic, then $\operatorname{Aut} \mathcal{P} \cong \prod_{i=1}^{k} \operatorname{Aut} P_i$.

Beta Values of Finite Groups

The beta value of a group G is the minimum number of points in a poset P needed to realize G. $\beta(G)$ is known for certain cyclic groups. Barmak proved that $\beta(Z_3) = 9$ [1], and we proved that $\beta(S_n) = n$ for all $n \in \mathbb{N}$. We did this through first assuming $\operatorname{Aut} P = S_n$ for some $n \geq 1$ and showing that $\operatorname{Aut} P \leq S_k$, where $k = |P|$. Therefore, $S_n = \operatorname{Aut} P \leq S_k$, $n \leq k$. So $\beta(S_n) \geq n$. Then we constructed a poset P with exactly n points such that $\operatorname{Aut} P = S_n$. So $\beta(S_n) = n$.

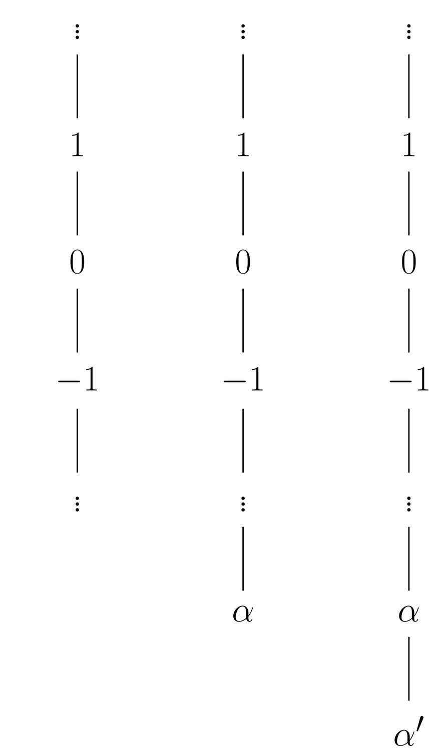
We have conjectured that $\beta(Z_p) = 3p$, and $\beta(Z_{p^k}) = 2p^k + p$ for any prime $p \geq 7$ by following Barmak's argument for $\beta(Z_3)$ [1].

Realizing All Finitely Generated Abelian Groups

Using the disjoint union theorem, we can realize all finitely generated abelian groups

$$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_k}$$

As a first example, we realize $\mathbb{Z}^3$. Consider $\mathcal{P}$, shown below:



First, we proved that $\operatorname{Aut} \mathbb{Z} \cong \mathbb{Z}$
The only automorphisms of $\mathbb{Z}$ are translations, $f(x) = x + n$
We showed that adding minimal nodes (such as α) does not change the automorphism group.
Therefore,
$\operatorname{Aut} \mathbb{Z} \cup \{\alpha\} \cong \operatorname{Aut} \mathbb{Z} \cup \{\alpha, \alpha'\} \cong \mathbb{Z}$
And so,

$$\operatorname{Aut} \mathcal{P} \cong \mathbb{Z}^3$$

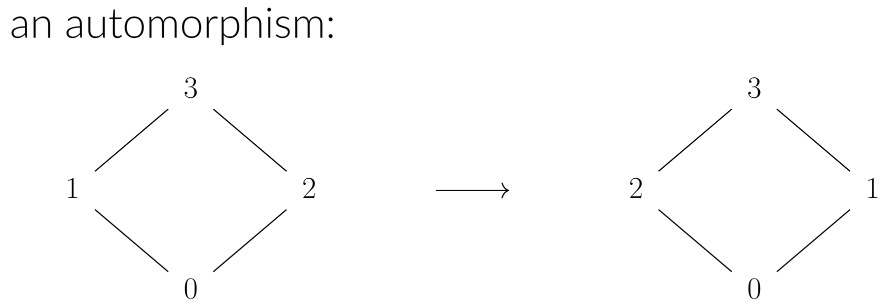
The same argument can be extended to realize $\mathbb{Z}^r$. Now, to get $\mathbb{Z}^r \times T$, we realize T by using Barmak and Minian's construction [2] and then we use the Disjoint Union Theorem to get the full group G. All together,

Theorem

Every finitely generated abelian group is realizable as the automorphism group of some poset

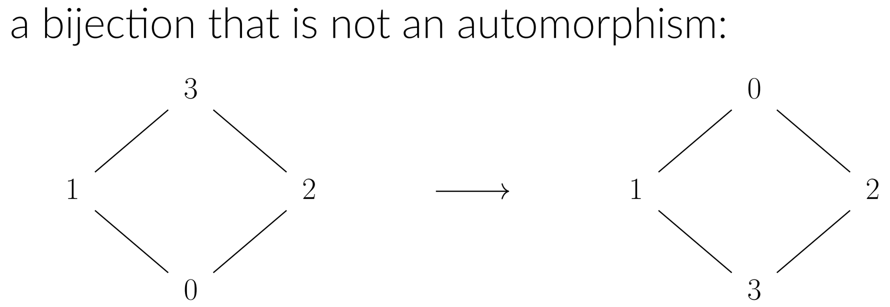
Python Code for Finding Automorphisms

Consider an automorphism:



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \end{array} \to \begin{array}{c|cccc} & 0 & 2 & 1 & 3 \\ \hline 0 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Consider a bijection that is not an automorphism:



$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 0 & 1 \end{array} \to \begin{array}{c|cccc} & 3 & 1 & 2 & 0 \\ \hline 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{array}$$

Figure 1. Automorphism Counter on GitHub

Other Results and Partial Progress

- If F_S is the free group on a nonempty finite set S, it is possible to turn it into a poset by declaring $x \leq_{F_S} y$ if and only if $y = xg_1 \cdots g_k$, where $g_i \in S \cup \{1_{F_S}\}$ for all i.
- If P is a poset, it is possible to add additional points to P to eliminate automorphisms of P.
- We improved the efficiency of our original Python code by considering heights of nodes instead of all possible bijections.

Future Work

- Show that all free groups are realizable.
- Calculate $\beta(Z_{p^k})$ for all primes p.
- Make Python code even more efficient.

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